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# NOTES ON PSEUDO-ANOSOVs WITH SMALL DILATATIONS COMING FROM THE MAGIC 3-MANIFOLD

EIKO KIN

## 1. INTRODUCTION

Let  $N$  be the exterior of the 3 chain link  $C_3$  (Figure 1) in the three sphere  $S^3$ . Gordon and Wu called  $N$  the *magic manifold*, because they found that  $N$  has many interesting non-hyperbolic fillings and this particular manifold plays a significant role for the study of non-hyperbolic fillings for cusped hyperbolic 3-manifolds. The magic manifold  $N$  is a hyperbolic surface bundles over the circle, and  $N$  has the smallest known volume among orientable 3-cusped hyperbolic 3-manifolds. Martelli and Petronio classified all the non-hyperbolic Dehn fillings of  $N$  in [18]. Let  $N(r)$  be the manifold obtained from  $N$  by Dehn filling one cusp along the slope  $r \in \mathbb{Q}$ . The Whitehead link exterior and the Whitehead sister link (i.e,  $(-2, 3, 8)$ -pretzel link) exterior are homeomorphic to  $N(1)$  and  $N(\frac{3}{-2})$  respectively. It was proved by Agol [2] that the smallest volume among orientable 2-cusped hyperbolic 3-manifold is achieved by either  $N(1)$  or  $N(\frac{3}{-2})$ . In the recent work of Gabai, Meyerhoff and Milley, the magic manifold  $N$  plays a central role for the minimizing problem on volumes of hyperbolic 3-manifolds. The main characters in this paper are manifolds  $N$ ,  $N(1)$ ,  $N(\frac{3}{-2})$  and  $N(\frac{1}{-2})$ . The last 2-cusped 3-manifold  $N(\frac{1}{-2})$  is homeomorphic to the exterior of the  $6_2^2$  link (Figure 1).

In [11, 12, 13, 14], we investigated the monodromies of fibrations of  $N$  extensively for the study of the minimal dilatations and their asymptotic behaviors. We found that  $N$  provides many interesting families of pseudo-Anosovs with small dilatations. In this paper, we give an expository account of results of [11, 12, 13, 14]. All the results in the paper are contained in those papers, and hence this paper has no new results. The purpose of this paper is to describe “places in  $N$ ” where the pseudo-Anosovs with the smallest dilatations or with the smallest known dilatations “live”. The main tool to do this is a fibered face of the Thurston norm ball for  $N$ .

Let  $\Sigma_{g,n}$  be an orientable surface of genus  $g$  with  $n$  punctures, and let  $\Sigma_g = \Sigma_{g,0}$  be a closed surface of genus  $g$ . We consider the mapping class group  $\text{Mod}(\Sigma)$  of  $\Sigma = \Sigma_{g,n}$ , that

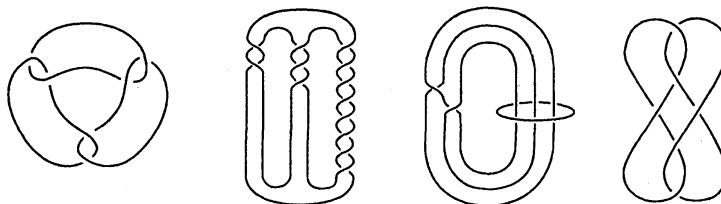


FIGURE 1. (from left to right) 3 chain link  $C_3$ ,  $(-2, 3, 8)$ -pretzel link, link  $6_2^2$ , Whitehead link.

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is the group of isotopy classes of orientation preserving homeomorphisms on  $\Sigma$ . According to the work of Nielsen and Thurston, elements of  $\text{Mod}(\Sigma)$  are classified into three types: periodic, reducible, pseudo-Anosov. The last type, pseudo-Anosovs have many interesting and rich properties. The hyperbolization theorem by Thurston asserts that  $\phi \in \text{Mod}(\Sigma)$  is pseudo-Anosov if and only if the mapping torus  $\mathbb{T}(\phi)$  of  $\phi$  is a hyperbolic 3-manifold with finite volume.

Each pseudo-Anosov  $\phi \in \text{Mod}(\Sigma)$  has a representative  $\Phi : \Sigma \rightarrow \Sigma$ , called a *pseudo-Anosov homeomorphism*, which satisfies the following: there exists a constant  $\lambda > 1$  and there exists a pair of transverse measured foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$  such that

$$\Phi(\mathcal{F}^s) = \frac{1}{\lambda} \mathcal{F}^s \text{ and } \Phi(\mathcal{F}^u) = \lambda \mathcal{F}^u.$$

The constant  $\lambda = \lambda(\Phi)$  is called the *dilatation* of  $\Phi$ , and  $\mathcal{F}^s, \mathcal{F}^u$  are called the *stable, unstable foliation* (or *invariant foliations*) of  $\Phi$ . It is known that  $\lambda(\Phi)$  does not depend on the choice of a pseudo-Anosov homeomorphism  $\Phi \in \phi$ , and hence the *dilatation*  $\lambda(\phi)$  of  $\phi$  is defined to be  $\lambda(\Phi)$ . We call the quantities

$$\text{ent}(\phi) = \log \lambda(\phi) \text{ and } \text{Ent}(\phi) = |\chi(\Sigma)| \log \lambda(\phi)$$

the *entropy* and *normalized entropy* of  $\phi$ , where  $\chi(\Sigma)$  is the Euler characteristic of  $\Sigma$ .

We fix  $\Sigma$  and consider the set of entropies defined on  $\Sigma$ :

$$\{\text{ent}(\phi) \mid \phi \in \text{Mod}(\Sigma) \text{ is pseudo-Anosov}\} \subset \mathbb{R}.$$

It is proved by Ivanov that this set is closed and discrete. In particular there exists a minimum. We denote by  $\delta(\Sigma) > 1$ , the minimal dilatation of pseudo-Anosov elements defined on  $\Sigma$ .

**Problem 1.1** (Minimal dilatation problem). *Determine the explicit value of  $\delta(\Sigma)$ . Identify a pseudo-Anosov element in  $\text{Mod}(\Sigma)$  which achieves  $\delta(\Sigma)$ .*

Let us set  $\delta_{g,n} = \delta(\Sigma_{g,n})$  and  $\delta_g = \delta_{g,0}$ . The explicit values of  $\delta_g$ 's are known for the only cases  $g = 1, 2$ . It is known by <sup>1</sup>Penner [22] that  $\log \delta_g \asymp \frac{1}{g}$ . After the work of Penner, several authors examined the asymptotic behaviors of the minimal dilatations on surfaces varying topology, see [9, 1, 13, 20, 10, 24] and Table 1(1st column).

Problem 1.1 has several aspects, and there are many related questions.

**Question 1.2** ([21] for (4)).

- (1) *Is a pseudo-Anosov element  $\phi \in \text{Mod}(\Sigma)$  which achieves  $\delta(\Sigma)$  unique up to conjugate?*
- (2) *Identify the hyperbolic fibered 3-manifold  $\mathbb{T}(\phi)$  of such a minimizer  $\phi$ .*
- (3) *What is the minimal polynomial of  $\delta(\Sigma)$ ? (Note: The dilatation  $\lambda(\phi)$  of a pseudo-Anosov  $\phi$  is known to be an algebraic integer.)*
- (4) *Do  $\lim_{g \rightarrow \infty} g \log \delta_g$ ,  $\lim_{g \rightarrow \infty} g \log \delta_g^+$ ,  $\lim_{n \rightarrow \infty} n \log \delta_{0,n}$  and  $\lim_{n \rightarrow \infty} n \log \delta_{1,n}$  exist? What are the values?*
- (5) *Given  $g \geq 2$ , does  $\lim_{n \rightarrow \infty} \frac{n \log \delta_{g,n}}{\log n}$  exist? What is its value?*

The smallest known upper bounds on Question 1.2(4)(5) are shown in Table 1(2nd column). We shall see that all families of pseudo-Anosovs  $\phi$ 's to give the upper bounds in Table 1(2nd column) 'come from'  $N$ . More precisely, these pseudo-Anosov mapping

<sup>1</sup>Let  $A_g$  and  $B_g$  be functions on  $g$ . We write  $A_g \asymp B_g$  if there exists a constant  $c$ , independent of  $g$ , such that  $\frac{A_g}{c} < B_g < cA_g$ .

TABLE 1. asymptotic behaviors of minimal dilatations.

asymptotic behaviors	upper bounds
$\log \delta_g \asymp 1/g$ [22]	$\limsup_{g \rightarrow \infty} g \log \delta_g \leq \log(\frac{3+\sqrt{5}}{2})$ [9, 1, 13]
$\log \delta_g^+ \asymp 1/g$ [20, 10]	$\limsup_{\substack{g \not\equiv 0 \pmod{6} \\ g \rightarrow \infty}} g \log \delta_g^+ \leq \log(\frac{3+\sqrt{5}}{2})$ [9, 11] $\limsup_{\substack{g \equiv 6 \pmod{12} \\ g \rightarrow \infty}} g \log \delta_g^+ \leq 2 \log \delta(D_5)$ [11]
$\log \delta_{0,n} \asymp 1/n$ [10]	$\limsup_{n \rightarrow \infty} n \log \delta_{0,n} \leq 2 \log(2 + \sqrt{3})$ [10, 12]
$\log \delta_{1,n} \asymp 1/n$ [24]	$\limsup_{n \rightarrow \infty} n \log \delta_{1,n} \leq 2 \log \delta(D_4)$ [11]
Given $g \geq 2$ , $\log \delta_{g,n} \asymp \frac{\log n}{n}$ [24]	$\limsup_{n \rightarrow \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2$ if $g$ enjoys (*) in Thm. 3.5 [14]

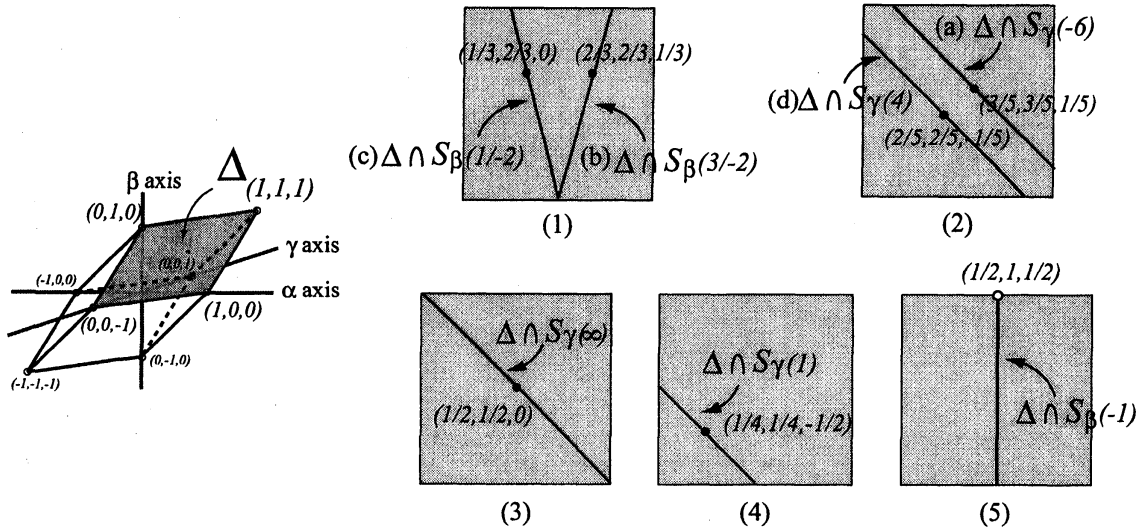


FIGURE 2. (left) Thurston norm ball  $U_N$  for  $N$ . (right) intersection of  $\Delta$  and linear section  $S_*(r)$ . (1)  $\Delta \cap S_\beta(\frac{1}{2})$  (see (c) in the figure) and  $\Delta \cap S_\beta(\frac{3}{2})$  (see (b) in the figure); (2)  $\Delta \cap S_\gamma(4)$  (see (d)) and  $\Delta \cap S_\gamma(-6)$  (see (a)); (3)  $\Delta \cap S_\gamma(\infty)$ ; (4)  $\Delta \cap S_\alpha(1) = \Delta \cap S_\beta(1) = \Delta \cap S_\gamma(1)$ ; (5)  $\Delta \cap S_\beta(-1)$ .

classes  $\phi$ 's have the following property: The mapping torus  $\mathbb{T}(\phi)$  is homeomorphic to  $N$ , or  $\mathbb{T}(\phi)$  is obtained from  $N$  by Dehn filling cusps along the boundary slopes of a fiber of  $N$ . (i.e,  $N$  is a *parent manifold* of  $\mathbb{T}(\phi)$ .)

Let  $\delta_g^+$  be the minimal dilatation of pseudo-Anosovs with orientable invariant foliations defined on  $\Sigma_g$ . (Obviously  $\delta_g \leq \delta_g^+$ .) The explicit value of  $\delta_g^+$  is known for all  $2 \leq g \leq 8$  except for  $g = 6$  [1, 9, 13, 16, 26]. (See Table 5(3rd column).) The minimal dilatation  $\delta(D_n)$  on an  $n$ -punctured disk  $D_n$  is determined for all  $3 \leq n \leq 8$  [7, 8, 15, 17]. (See Table 10(3rd column).) These minimizers come from  $N$  in the same sense as above.

The paper is organized as follows: In Section 2, we first review the fibered face theory which is quite useful to find families of pseudo-Anosovs with small dilatations. Next, we

describe the properties of fibrations on both  $N$  and manifolds  $N(r)$ 's. In Section 3, we examine the asymptotic behaviors of minimal dilatations given in Table 1. Especially we explain how the constants in the upper bounds of Table 1 (2nd column) appear. These constants are related to an invariant “min Ent” of hyperbolic surface bundles over the circle. Figure 2(left) shows the Thurston norm ball of  $N$ . A particular fibered face  $\Delta$  is shaded in the figure. By using Figure 2(right), we shall illustrate places in  $N$  where the pseudo-Anosovs with the smallest dilatations or with the smallest known dilatations live. (For the definition of the linear sections  $S_\beta(r)$  etc, see Section 2.2. See also Figure 3.) We conclude the paper with conjectures and questions.

## 2. PRELIMINARIES

**2.1. Basic facts on fibered face theory.** Let  $M$  be an oriented, hyperbolic 3-manifold possibly with boundary  $\partial M$ . We recall the Thurston norm  $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \rightarrow \mathbb{R}$ . See [23] for more details. The Thurston norm  $\|\cdot\|$  has the property such that for any integral class  $a \in H_2(M, \partial M; \mathbb{R})$ ,

$$\|a\| = \min_F \{-\chi(F)\},$$

where the minimum is taken over all oriented surfaces  $F$  embedded in  $M$ , satisfying  $a = [F]$ , with no components of non-negative Euler characteristic. The surface  $F$  which realizes this minimum is called a *minimal representative* of  $a$ , and it is denoted by  $F_a$ . For a rational number  $r$  and an integral class  $a \in H_2(M, \partial M; \mathbb{R})$ ,  $\|ra\|$  is defined to be  $\|ra\| = |r|\|a\|$ . The norm  $\|\cdot\|$  defined on rational classes admits a unique continuous extension to  $H_2(M, \partial M; \mathbb{R})$  which is linear on the ray through the origin. The unit ball  $U_M = \{a \in H_2(M, \partial M; \mathbb{R}) \mid \|a\| \leq 1\}$  is a compact, convex polyhedron.

Suppose that  $M$  is a surface bundle over the circle. We now recall Thurston's description of the relation between  $\|\cdot\|$  and fibrations of  $M$ . Let  $\Omega$  be a top dimensional face on  $\partial U_M$ . We denote the cone over  $\Omega$  with the origin by  $C_\Omega$ , and denote its interior by  $\text{int}(C_\Omega)$ . In [23], Thurston proved that if we let  $F$  be a fiber of a fibration of  $M$ , then there exists a top dimensional face  $\Omega$  such that  $[F]$  is an integral class of  $\text{int}(C_\Omega)$ . On the other hand, for *any* integral class  $a \in \text{int}(C_\Omega)$ , a minimal representative  $F_a$  becomes a fiber of the fibration associated to  $a$ . For this reason, such a face  $\Omega$  is called a *fibered face* and an integral class  $a \in \text{int}(C_\Omega)$  is called a *fibered class*. This property tells us that if  $M$  is a hyperbolic 3-manifold which is a surface bundle over the circle having the second Betti number more than 1, then it admits an infinite family of fibrations.

If a fibered class  $a \in \text{int}(C_\Omega)$  is primitive, then the fibration associated to  $a$  has a connected fiber represented by  $F_a$ . Since  $M$  is hyperbolic, the mapping class  $\phi_a = [\Phi_a]$  of the monodromy  $\Phi_a : F_a \rightarrow F_a$  is pseudo-Anosov. The *dilatation*  $\lambda(a)$  and *entropy*  $\text{ent}(a) = \log \lambda(a)$  are defined as the dilatation  $\lambda(\phi_a)$  and entropy  $\text{ent}(\phi_a)$  of  $\phi_a$  respectively.

We turn to the work of Fried, Matsumoto and McMullen. The entropy defined on primitive fibered classes is extended to rational classes as follows: For a rational number  $r$  and a primitive fibered class  $a$ , the entropy  $\text{ent}(ra)$  is defined by  $\frac{1}{|r|}\text{ent}(a)$ . Let  $\text{int}(C_\Omega(\mathbb{Q}))$  (resp.  $\text{int}(C_\Omega(\mathbb{Z}))$ ) be the set of rational classes (resp. integral classes) in  $\text{int}(C_\Omega)$ . Fried proved that  $\frac{1}{\text{ent}} : \text{int}(C_\Omega(\mathbb{Q})) \rightarrow \mathbb{R}$  is concave [6], and in particular  $\text{ent} : \text{int}(C_\Omega(\mathbb{Q})) \rightarrow \mathbb{R}$  admits a unique continuous extension

$$\text{ent} : \text{int}(C_\Omega) \rightarrow \mathbb{R}.$$

Moreover, Fried proved the following: The restriction of  $\text{ent}$  to the open fibered face  $\text{int}(\Omega)$  has the property such that  $\text{ent}(a)$  goes to  $\infty$  as  $a \in \text{int}(\Omega)$  goes to a point on  $\partial\Omega$ . Thus we have a continuous function

$$\text{Ent} = \|\cdot\| \text{ent}(\cdot) : \text{int}(C_\Omega) \rightarrow \mathbb{R}.$$

We call  $\text{Ent}(a)$  the *normalized entropy* of  $a \in \text{int}(C_\Omega)$ . By definition of  $\text{ent}$ , we see that  $\text{Ent}$  is constant on each ray in  $\text{int}(C_\Omega)$  through the origin. McMullen developed a theory of the *Teichmüller polynomial*  $P_\Omega$  for a fibered face  $\Omega$  of hyperbolic surface bundles over the circle, from which one can compute  $\lambda(a)$  of each  $a \in \text{int}(C_\Omega)$ , see [21].

By Matsumoto [19] and by McMullen [21], it was proved that  $\frac{1}{\text{ent}}$  on  $\text{int}(\Omega)$  is strictly concave. This implies that  $\text{ent}$  is strictly convex on  $\text{int}(\Omega)$  because  $\text{ent}$  is positive valued. Since  $\|\cdot\|$  is constant ( $= 1$ ) on a fibered face  $\Omega$ , the normalized entropy  $\text{Ent}$  is strictly convex on  $\text{int}(\Omega)$ . Thus  $\text{Ent}|_{\text{int}(\Omega)} : \text{int}(\Omega) \rightarrow \mathbb{R}$  has a minimum at a unique point in  $\text{int}(\Omega)$ . In other words,  $\text{Ent} : \text{int}(C_\Omega) \rightarrow \mathbb{R}$  admits a minimum at a unique ray through the origin. We denote this minimum by  $\min \text{Ent}(M, \Omega)$ . We also denote by  $\min_\Omega \text{Ent}(M)$ ,  $\min_\Omega \{\min \text{Ent}(M, \Omega)\}$ , where  $\Omega$  is taken over all fibered faces for  $M$ .

**2.2. Properties of fibrations on the magic manifold.** In this section, we collect particular properties on  $N$  which are needed in the rest of the paper.

Let  $K_\alpha, K_\beta$  and  $K_\gamma$  be the components of the 3 chain link  $C_3$ . They bound the oriented disks  $F_\alpha, F_\beta$  and  $F_\gamma$  with 2 holes. Let us set  $\alpha = [F_\alpha], \beta = [F_\beta], \gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$ . The Thurston (unit) ball  $U_N$  is the parallelepiped with vertices  $\pm\alpha, \pm\beta, \pm\gamma, \pm(\alpha + \beta + \gamma)$ , see Figure 2(left). Every top dimensional face on  $\partial U_N$  is a fibered face by the symmetries of  $H_2(N, \partial N)$ . The set  $\{\alpha, \beta, \gamma\}$  is a basis of  $H_2(N, \partial N; \mathbb{Z})$ , and  $x\alpha + y\beta + z\gamma \in H_2(N, \partial N)$  is denoted by  $(x, y, z)$ .

We denote by  $T_\alpha$ , the torus which is the boundary of a regular neighborhood of  $K_\alpha$ . We define the tori  $T_\beta$  and  $T_\gamma$  in the same manner. For a primitive integral class  $a = (x, y, z) \in H_2(N, \partial N)$ , let us set  $\partial_\alpha F_a = \partial F_a \cap T_\alpha$  which consists of the parallel simple closed curves on  $T_\alpha$ . We define  $\partial_\beta F_a$  and  $\partial_\gamma F_a$  in the same manner.

Pick a fibered face  $\Delta$  on  $\partial U_N$  as in Figure 2(left) with vertices  $(1, 0, 0), (1, 1, 1), (0, 1, 0)$  and  $(0, 0, -1)$ . The open face  $\text{int}(\Delta)$  is written by

$$\text{int}(\Delta) = \{(x, y, z) \mid x + y - z = 1, x > 0, y > 0, x > z, y > z\}.$$

The Thurston norm of  $(x, y, z) \in \text{int}(C_\Delta)$  is given by  $x + y - z$ .

**Proposition 2.1** ([11]). *Let  $a = (x, y, z)$  be a primitive fibered class in  $\text{int}(C_\Delta)$ .*

- (1) *The number of the boundary components  $\#(\partial F_a)$  of  $F_a$  is given by*

$$\#(\partial F_a) = \gcd(x, y + z) + \gcd(y, z + x) + \gcd(z, x + y),$$

*where  $\gcd(0, w)$  is defined by  $|w|$ . More precisely*

$$\#(\partial_\alpha F_a) = \gcd(x, y + z), \#(\partial_\beta F_a) = \gcd(y, z + x), \#(\partial_\gamma F_a) = \gcd(z, x + y).$$

- (2)  *$\lambda(a) = \lambda_{(x, y, z)}$  equals the largest real root of*

$$f_{(x, y, z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1,$$

*where  $f_{(x, y, z)}(t)$  is the specialization of the Teichmüller polynomial  $P_\Delta$  at  $(x, y, z)$ .*

- (3) *The inverse  $\Phi_{(x, y, z)}^{-1}$  of  $\Phi_{(x, y, z)} : F_{(x, y, z)} \rightarrow F_{(x, y, z)}$  is conjugate to the monodromy  $\Phi_{(y, x, z)} : F_{(y, x, z)} \rightarrow F_{(y, x, z)}$  of the fibration on  $N$  associated to  $(y, x, z) \in \text{int}(C_\Delta)$ . In particular  $\lambda_{(x, y, z)} = \lambda_{(y, x, z)}$ .*

- (4)  $\min \text{Ent}(N) = \min \text{Ent}(N, \Delta) = \text{Ent}((\frac{1}{2}, \frac{1}{2}, 0)) = 2 \log(2 + \sqrt{3}) \approx 2.6339$ .  
 (5) *The stable foliation  $\mathcal{F}_a$  of  $\Phi_a : F_a \rightarrow F_a$  has the property such that each component of  $\partial_\alpha F_a$ ,  $\partial_\beta F_a$  and  $\partial_\gamma F_a$  has  $\frac{x}{\gcd(x, y+z)}$  prongs,  $\frac{y}{\gcd(y, x+z)}$  prongs and  $\frac{x+y-2z}{\gcd(z, x+y)}$  prongs respectively. Moreover  $\mathcal{F}_a$  does not have singularities in the interior of  $F_a$ .*  
 (6)  $\mathcal{F}_a$  is orientable if and only if  $x$  and  $y$  are even and  $z$  is odd.

We see that the slope of  $\partial_\alpha F_a$  (resp.  $\partial_\beta F_a$ ,  $\partial_\gamma F_a$ ) is given by  $b_\alpha(a) = \frac{y+z}{-x}$  (resp.  $b_\beta(a) = \frac{z+x}{-y}$ ,  $b_\gamma(a) = \frac{x+y}{-z}$ ). We call each of  $b_\alpha(a)$ ,  $b_\beta(a)$ ,  $b_\gamma(a)$  the *boundary slope* of  $a$ .

By using the formula in Proposition 2.1, we recover the similar formula for *any* primitive fibered classes  $a \in H_2(N, \partial N)$ . This is because there is a homeomorphism  $h : (S^3, \mathcal{C}_3) \rightarrow (S^3, \mathcal{C}_3)$  which sends  $K_\alpha$ ,  $K_\beta$ ,  $K_\gamma$  to  $K_\beta$ ,  $K_\gamma$ ,  $K_\alpha$  respectively, and  $H_2(N, \partial N)$  has symmetries by the isomorphism  $h_* : H_2(N, \partial N) \rightarrow H_2(N, \partial N)$  of order 3 induced from  $h$ .

It is known by [18] that  $N(r)$  is hyperbolic if and only if  $r \in \mathcal{Hyp} = \mathbb{Q} \setminus \{-3, -2, -1, 0\}$ . We now recall the description of fibered classes of the hyperbolic Dehn filling  $N(r)$ 's. Let  $N(r)$  be the manifold obtained from  $N$  by Dehn filling the cusp specified by, say  $T_\beta$ , along the slope  $r \in \mathbb{Q}$  or  $r = \frac{1}{0} (= \infty)$ . Then, there exists a natural injection

$$(1) \quad \iota_\beta : H_2(N(r), \partial N(r)) \rightarrow H_2(N, \partial N)$$

whose image equals the linear section  $S_\beta(r)$ , where

$$S_\beta(r) = \{(x, y, z) \in H_2(N, \partial N) \mid -ry = z + x\},$$

see [11, Proposition 2.11]. Choose  $r \in \mathcal{Hyp}$ , and assume that  $a \in S_\beta(r) = \text{Im } \iota_\beta$  is a fibered class in  $H_2(N, \partial N)$ . Then,  $\bar{a} = \iota_\beta^{-1}(a) \in H_2(N(r), \partial N(r))$  is also a fibered class of  $N(r)$ . We sometimes denote  $N(r)$  by  $N_\beta(r)$  when we need to specify the cusp which is filled.

Similarly, when  $N(r)$  is the manifold obtained from  $N$  by Dehn filling the cusp specified by  $T_\alpha$  or  $T_\gamma$  along the slope  $r$ , one has natural injections,

$$\begin{aligned} \iota_\alpha : H_2(N(r), \partial N(r)) &\rightarrow H_2(N, \partial N), \\ \iota_\gamma : H_2(N(r), \partial N(r)) &\rightarrow H_2(N, \partial N) \end{aligned}$$

such that their images are

$$\begin{aligned} S_\alpha(r) &= \{(x, y, z) \in H_2(N, \partial N) \mid -rx = y + z\}, \\ S_\gamma(r) &= \{(x, y, z) \in H_2(N, \partial N) \mid -rz = x + y\}. \end{aligned}$$

We may denote by  $N_\alpha(r)$  or  $N_\gamma(r)$ , the manifold  $N(r)$  in this case. This description enables us to compute the Thurston norm of  $N(r)$ , especially the Thurston unit ball and fibered faces. For more detailed computation, see [11]. Figure 3 illustrates the intersection of the Thurston norm ball  $U_N$  and the linear section  $S_*(r)$ ,  $*$   $\in$   $\{\alpha, \beta, \gamma\}$ .

**Remark 2.2** (Lemmas 3.28 and 5.2 in [11]). Take  $r \in \mathcal{Hyp}$ , and let  $\bar{a} \in H_2(N(r), \partial N(r))$  be a primitive integral class. If  $r \neq 1$ , then  $\#(\partial F_{\bar{a}})$  is bounded by a constant from above which depends on  $r$ . On the other hand, in the case  $r = 1$ , the genus of  $F_{\bar{a}}$  is always equal to 1, and hence there exists no upper bound of  $\#(\partial F_{\bar{a}})$ .

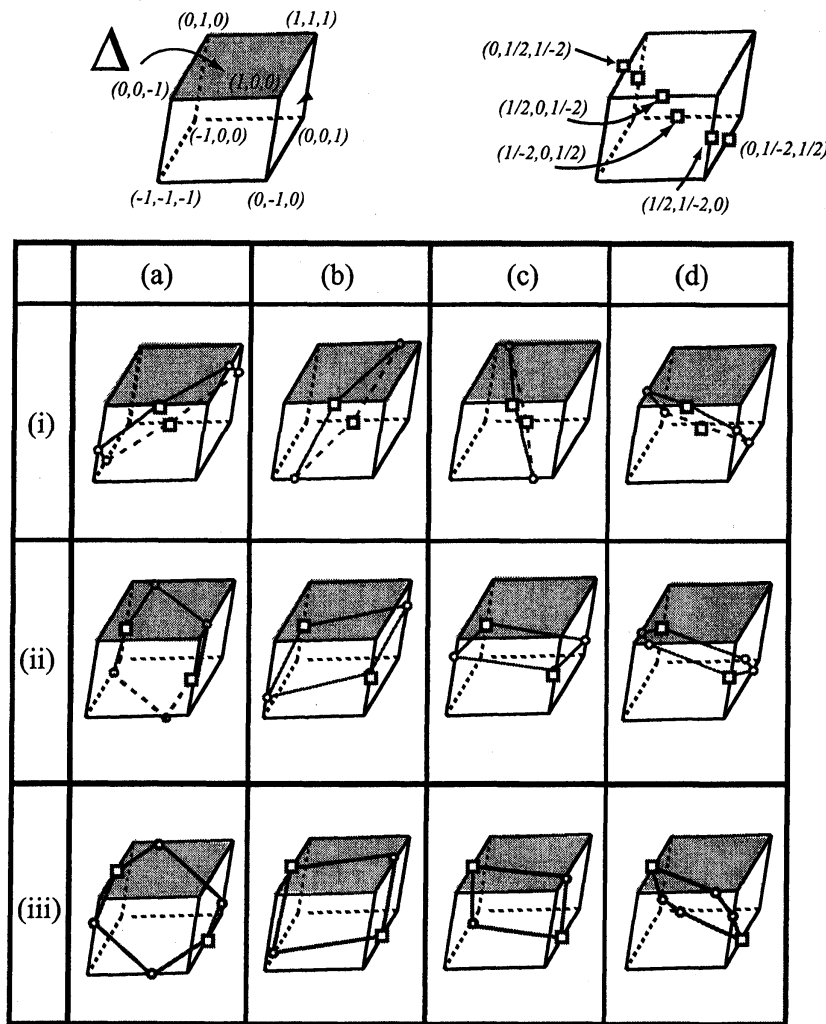


FIGURE 3. 1st row (i)  $U_N \cap S_\beta(r)$ , 2nd row (ii)  $U_N \cap S_\gamma(r)$  and 3rd row (iii)  $U_N \cap S_\alpha(r)$ . [(a)  $r \in (-\infty, -2)$ , (b)  $r \in (-2, -1)$ , (c)  $r \in (-1, 0)$ , (d)  $r \in (0, \infty)$ .] [the fibered face  $\Delta$  is shaded in the figure.]

**2.3. Entropy equivalence on the manifolds  $N(r)$ 's.** The notation “entropy equivalence” on fibered 3-manifolds was introduced in [11]. By using this equivalence relation, we will see in Theorem 2.3 that there are infinitely many entropy equivalent pairs among  $N(r)$ 's. The particular pair is  $N(\frac{3}{-2})$  and  $N(\frac{1}{-2})$ . They are not homeomorphic to each other, but they have common properties on the normalized entropy.

We say that 3-manifolds  $M$  and  $M'$  are *Thurston norm equivalent*, denoted by  $M \underset{T}{\sim} M'$ , if there exists an isomorphism  $f : H_2(M, \partial M; \mathbb{Z}) \rightarrow H_2(M', \partial M'; \mathbb{Z})$  which preserves the Thurston norm, i.e.,  $\|a\| = \|f(a)\|$  for any  $a \in H_2(M, \partial M; \mathbb{Z})$ . We call such  $f$  the *Thurston norm preserving isomorphism*.

Let  $(M, \Omega)$  and  $(M', \Omega')$  be pairs of 3-manifolds  $M$ ,  $M'$  and their fibered faces  $\Omega$ ,  $\Omega'$  respectively. Possibly  $M \simeq M'$ . Then  $(M, \Omega)$  and  $(M', \Omega')$  are *entropy equivalent*,



denoted by  $(M, \Omega) \underset{\text{ent}}{\sim} (M', \Omega')$ , if there exists a Thurston norm preserving isomorphism  $f : H_2(M, \partial M; \mathbb{Z}) \rightarrow H_2(M', \partial M'; \mathbb{Z})$  satisfying the following.

- $a \in \text{int}(C_\Omega(\mathbb{Z}))$  if and only if  $f(a) \in \text{int}(C_{\Omega'}(\mathbb{Z}))$ .
- $\text{ent}(a) = \text{ent}(f(a))$  for any  $a \in \text{int}(C_\Omega(\mathbb{Z}))$ .

The second bullet implies that  $\text{ent}(a) = \text{ent}(f(a))$  for any  $a \in \text{int}(C_\Omega)$  since  $\text{ent} : \text{int}(C_\Omega(\mathbb{Q})) \rightarrow \mathbb{R}$  admits a unique continuous extension. Thus if  $(M, \Omega) \underset{\text{ent}}{\sim} (M', \Omega')$ , then  $\min \text{Ent}(M, \Omega) = \min \text{Ent}(M', \Omega')$ .

Fibered 3-manifolds  $M$  and  $M'$  are *entropy equivalent*, denoted by  $M \underset{\text{ent}}{\sim} M'$ , if there exists a Thurston norm preserving isomorphism  $f : H_2(M, \partial M; \mathbb{Z}) \rightarrow H_2(M', \partial M'; \mathbb{Z})$  satisfying the following.

- $a \in H_2(M, \partial M; \mathbb{Z})$  is a fibered class if and only if  $f(a) \in H_2(M', \partial M'; \mathbb{Z})$  is a fibered class.
- Given a fibered face  $\Omega$  of  $M$ , we have  $\text{ent}(a) = \text{ent}(f(a))$  for any  $a \in \text{int}(C_\Omega(\mathbb{Z}))$ .

If  $M \underset{\text{ent}}{\sim} M'$ , then  $\min \text{Ent}(M) = \min \text{Ent}(M')$ .

We turn to the manifolds  $N(r)$ 's. Let  $p \in \mathbb{N}$  and  $q \in \mathbb{Z}$  be coprime such that  $r = \frac{p}{q} \in \text{Hyp}$ . Then  $N(r)$  has two kinds of fibered faces, *A-face* and *S-face*, see [11, Section 2.5]. When  $r \in (-2, 0)$ , the Thurston norm ball of  $N(r)$  is a parallelogram and every fibered face is an *A-face*. When  $r \in (-\infty, -2) \cup (0, \infty)$  such that  $|q| \neq 1$  (resp.  $|q| = 1$ ), the Thurston norm ball for  $N(r)$  is a hexagon (resp. rectangle) having two *S-faces* and four *A-faces* (resp. having two *S-faces* and two *A-faces*). cf. Figure 3. One can show that any two *S-faces* of  $N(r)$  are entropy equivalent, and any two *A-faces* of  $N(r)$  are entropy equivalent [11, Lemma 2.22]. In the case  $r = 1$ , by the symmetry of the Whitehead link exterior  $N(1)$  itself, one can see that an *S-face* of  $N(1)$  and an *A-face* of  $N(1)$  are entropy equivalent [11, Proposition 3.26]. Moreover the fibered class  $(1, 1, -2) \in H_2(N_\gamma(1), \partial N_\gamma(1))$  achieves  $\min \text{Ent}(N(1))$  [11, Corollary 3.27];

$$\min \text{Ent}(N(1)) = \text{Ent}(\overline{(1, 1, -2)}) = 2 \log \delta(D_4) \approx 1.6628.$$

An *S-face* of  $N(r)$  may not be entropy equivalent to an *A-face* of  $N(r)$  for other  $r$ .

**Theorem 2.3** (Theorem 2.26 in [11]). *Let  $p \in \mathbb{N}$  and  $q \in \mathbb{Z}$  be as above.*

- (1) *Suppose that  $\frac{p}{q} \in (-\infty, -2)$  and  $p + 2q \neq 1$ . Then  $(N(\frac{p}{q}), \Omega_S) \underset{\text{ent}}{\sim} (N(\frac{2q+p}{-q}), \Omega_S)$ .*
- (2) *Suppose that  $\frac{p}{q} \in (-\infty, -1)$  and  $|q| \neq 1$ . Then  $(N(\frac{p}{q}), \Omega_A) \underset{\text{ent}}{\sim} (N(\frac{-2q-p}{q}), \Omega_A)$ .*
- (3) *Suppose that  $\frac{p}{q} \in (-\infty, -1)$ ,  $p + 2q \neq 1$  and  $|q| \neq 1$ . Then  $N(\frac{p}{q}) \underset{\text{ent}}{\sim} N(\frac{-2q-p}{q})$ .*

In Proposition 2.4, we will see that the entropy function on  $N$  has symmetries. This property is a key for the proof of Theorem 2.3. By Theorem 2.3,

$$(N(-6), \Omega_S) \underset{\text{ent}}{\sim} (N(4), \Omega_S) \text{ and } N(\frac{3}{-2}) \underset{\text{ent}}{\sim} N(\frac{1}{-2}).$$

Table 2 exhibits the computation of  $\min \text{Ent}$  for these manifolds. Readers may notice that we encountered these numbers  $\min \text{Ent}$  in the upper bounds of Table 1(2nd column). It turns out that the both  $\min \text{Ent}(N(r), \Omega_A)$  for  $r = \frac{3}{-2}, \frac{1}{-2}$  and  $\min \text{Ent}(N(r), \Omega_S)$  for  $r = -6, 4$  are achieved by fibered classes for  $N(r)$ , see Table 2. The topological types of the fibers are also shown in the table. (e.g.  $\bar{a} = (3, 3, 1) \in H_2(N_\gamma(-6), \partial N_\gamma(-6))$  achieves  $\min \text{Ent}(N(-6), \Omega_S)$ , and  $F_{\bar{a}} \simeq \Sigma_{2,2}$ .)

TABLE 2.  $\min \text{Ent}$  for some  $N(r)$ 's. [note: the technique in [11] does not work for the computation of  $\min \text{Ent}(N(r), \Omega_A)$  in the case  $r = -6, 4$ .]

$N(r)$	$\min \text{Ent}(N(r), \Omega_S)$ fibered class, fiber	$\min \text{Ent}(N(r), \Omega_A)$ fibered class, fiber	$\min \text{Ent}(N(r))$
$N(\frac{3}{-2})$	none	$\frac{2 \log(\frac{3+\sqrt{5}}{2})}{(2, 2, 1), \Sigma_{1,2}}$	$2 \log(\frac{3+\sqrt{5}}{2}) \approx 1.9248$
$N(\frac{1}{-2})$	none	$\frac{2 \log(\frac{3+\sqrt{5}}{2})}{(1, 2, 0), \Sigma_{0,4}}$	$2 \log(\frac{3+\sqrt{5}}{2}) \approx 1.9248$
$N(-6)$	$\frac{4 \log \delta(D_5)}{(3, 3, 1), \Sigma_{2,2}}$	?	$\leq 4 \log \delta(D_5) \approx 2.1740$
$N(4)$	$\frac{4 \log \delta(D_5)}{(2, 2, -1), \Sigma_{2,2}}$	?	$\leq 4 \log \delta(D_5) \approx 2.1740$

**2.4. Mysterious symmetries of entropy function on the magic manifold.** The entropy function on  $N$  has mysterious symmetries not coming from the symmetries of  $N$  itself, which we will recall below.

We take  $(x, y, z) \in \Delta$ . (Hence  $x + y - z = 1$ .) Let us denote  $(x, y, z)$  by  $[x, y]$ . Then the open face  $\text{int}(\Delta)$  is written by

$$\text{int}(\Delta) = \{[x, y] \mid 0 < x < 1, 0 < y < 1\}.$$

On the other hand if  $(x, y, z) \in \text{int}(C_\Delta)$ , then

$$(y - z, y, y - x), (y - z, x - z, -z), (x, x - z, x - y) \in \text{int}(C_\Delta).$$

These four classes have the same Thurston norm. Intriguingly, they have the same dilatation!

**Proposition 2.4** (Lemma 2.5 in [11]). *The four classes*

$$(x, y, z), (y - z, y, y - x), (y - z, x - z, -z), (x, x - z, x - y) \in \text{int}(C_\Delta)$$

*have the same dilatation. In particular,*

$$\left[\frac{x}{x+y-z}, \frac{y}{x+y-z}\right], \left[\frac{y-z}{x+y-z}, \frac{y}{x+y-z}\right], \left[\frac{y-z}{x+y-z}, \frac{x-z}{x+y-z}\right], \left[\frac{x}{x+y-z}, \frac{x-z}{x+y-z}\right] \in \text{int}(\Delta)$$

*have the same dilatation. (See Figure 4(left).)*

We note that the topological types of  $F_{(x,y,z)}$ ,  $F_{(y-z,y,y-x)}$ ,  $F_{(y-z,x-z,-z)}$ ,  $F_{(x,x-z,x-y)}$  may be different. (e.g.  $F_{(6,5,4)} \simeq \Sigma_{0,9}$ ,  $F_{(1,5,-1)} \simeq \Sigma_{1,7}$ ,  $F_{(1,2,-4)} \simeq \Sigma_{3,3}$  and  $F_{(6,2,1)} \simeq \Sigma_{2,5}$ .) On the other hand by Proposition 2.1(3), any two classes  $a = [x, y], \tilde{a} \in [y, x] \in \text{int}(\Delta)$  have the same dilatation. This together with Proposition 2.4 says that 8 classes  $b_0, \tilde{b}_0, \dots, b_3, \tilde{b}_3 \in \text{int}(\Delta)$  as in Figure 4(right) have the same dilatation.

### 3. ASYMPTOTIC BEHAVIORS OF MINIMAL DILATATIONS

**3.1. Sequence  $\{\delta_g\}_{g \geq 2}$ .** Let  $\Phi : F \rightarrow F$  be the monodromy of a fibration on  $N$ , and let  $\phi = [\Phi]$ . Then the fibration extends naturally to a fibration on the closed manifold obtained from  $N$  by Dehn filling three cusps along boundary slopes of  $F$ . Also,  $\Phi$  extends to the monodromy  $\hat{\Phi} : \hat{F} \rightarrow \hat{F}$  of the extended fibration, where the extended fiber  $\hat{F}$  is obtained from  $F$  by filing holes. Suppose that the stable foliation  $\mathcal{F}$  of  $\Phi$  has the property

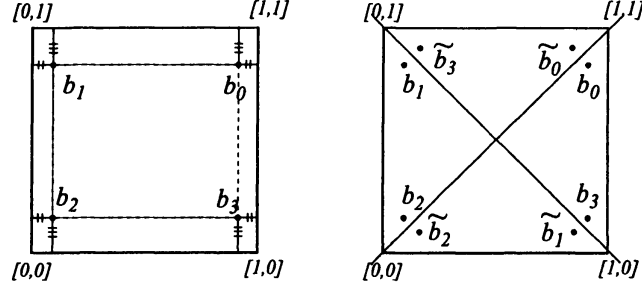


FIGURE 4.  $b_0 = [\frac{x}{x+y-z}, \frac{y}{x+y-z}]$ ,  $b_1 = [\frac{y-z}{x+y-z}, \frac{y}{x+y-z}]$ ,  $b_2 = [\frac{y-z}{x+y-z}, \frac{x-z}{x+y-z}]$ ,  $b_3 = [\frac{x}{x+y-z}, \frac{x-z}{x+y-z}] \in \text{int}(\Delta)$  and  $\tilde{b}_i \in \text{int}(\Delta)$ .

such that any boundary component of  $F$  has no 1 prong. Then  $\mathcal{F}$  extends canonically to the stable foliation  $\widehat{\mathcal{F}}$  of  $\widehat{\Phi}$ , and  $\widehat{\phi} = [\widehat{\Phi}]$  becomes pseudo-Anosov (including Anosov) with the same dilatation as that of  $\phi$ . We consider the set  $\mathcal{M}$  of (pseudo-Anosov) mapping classes coming from fibrations of  $N$  with this condition.

Now, let us denote by  $\widehat{\mathcal{M}}$ , the set of extensions  $\widehat{\phi}$  of  $\phi \in \mathcal{M}$  defined on the closed surfaces. Let  $\widehat{\delta}_g$  be the minimum among dilatations of elements in  $\widehat{\mathcal{M}} \cap \text{Mod}(\Sigma_g)$ . Clearly  $\delta_g \leq \widehat{\delta}_g$ . The equality holds when  $g = 2$ . (In fact  $\delta_2$  is achieved by  $\widehat{\phi}_a \in \widehat{\mathcal{M}} \cap \text{Mod}(\Sigma_2)$  when  $a = (2, 2, -1)$  or  $(2, 6, 1)$ .)

The set  $\mathcal{M}$  is large in the following sense. For any  $r \in \text{Hyp} \setminus \{1\}$ , there exist infinitely many primitive fibered classes  $a_n = a_n(r) \in S_\beta(r)$  such that  $\phi_{a_n} \in \mathcal{M}$  and the genus of  $F_{a_n}$  goes to  $\infty$  as  $n$  goes to  $\infty$ . In [11], we addressed Question 1.2(4) (about the asymptotic behavior of  $g \log \delta_g$ ) in  $\widehat{\mathcal{M}}$ .

**Theorem 3.1** (Theorem 1.4 in [11]).

- (1) We have  $\lim_{g \rightarrow \infty} g \log \widehat{\delta}_g = \log(\frac{3+\sqrt{5}}{2})$ .
- (2) For large  $g$ ,  $\widehat{\delta}_g$  is achieved by the monodromy of some  $\Sigma_g$ -bundle over the circle obtained from either  $N(\frac{3}{-2})$  or  $N(\frac{1}{-2})$  by Dehn filling both cusps.

More precisely, one can show the following: For large  $g$  such that  $g \equiv 0, 1, 5, 6, 7, 9 \pmod{10}$  (resp. such that  $g \equiv 3, 8 \pmod{10}$ ),  $\widehat{\delta}_g$  is achieved by the monodromy of some  $\Sigma_g$ -bundle over the circle obtained from  $N(\frac{3}{-2})$  (resp.  $N(\frac{1}{-2})$ ) by Dehn filling both cusps, see [11, Remark 3.18].

Table 3 shows the fibered class  $(x, y, z) \in H_2(N, \partial N)$  which achieves  $\widehat{\delta}_g$  for large  $g$  and the polynomial  $f_{(x,y,z)}(t)$ . Notice that such a fibered class  $(x, y, z)$  is in either  $\text{int}(C_\Delta) \cap S_\beta(\frac{3}{-2})$  or  $\text{int}(C_\Delta) \cap S_\beta(\frac{1}{-2})$ , see (1) in Section 2.2. Its projective class  $(x', y', z') \in \text{int}(\Delta)$  goes to the projective class of either  $(2, 2, 1)$  or  $(1, 2, 0)$  as the Thurston norm  $\|(x, y, z)\|$  goes to  $\infty$ , see Figure 2(1).

For small  $g$ , our upper bound of  $\delta_g$  is given by the brute computation, see Table 4. We note that in the case  $g = 8, 13$ ,  $\widehat{\delta}_g$  is not achieved by the monodromy of any  $\Sigma_g$ -bundle over the circle obtained from either  $N(\frac{3}{-2})$  or  $N(\frac{1}{-2})$  by Dehn filling [13, Proposition 4.37].

We describe the outline of the proof of Theorem 3.1(1). It is known that  $N(-4) \simeq N(\frac{3}{-2})$ , see [18]. We recall:

**Claim 3.2** (Theorem 1.5 in [13]). *Let  $r \in \{\frac{3}{-2}, \frac{1}{-2}, 2\}$ . For each  $g \geq 3$ , there exist  $\Sigma_g$ -bundles over the circle obtained from  $N(r)$  by Dehn filling both cusps along boundary slopes of fibers of  $N(r)$ . Among them, there exist monodromies  $\Phi_g(r) : \Sigma_g \rightarrow \Sigma_g$  of the fibrations such that*

$$\lim_{g \rightarrow \infty} g \log \lambda(\Phi_g(r)) = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

Let  $a_g$  be a primitive fibered class of  $H_2(N, \partial N)$  such that  $\phi_{a_g} \in \mathcal{M}$  and  $\widehat{\delta}_g$  is achieved by  $\widehat{\phi}_{a_g} \in \widehat{\mathcal{M}} \cap \text{Mod}(\Sigma_g)$ . Since  $N(1)$  has no fiber of genus greater than 1,  $a_g$  does not have a boundary slope 1 for  $g \geq 2$ . By the analysis of  $\min \text{Ent}(N(r), \Omega)$  (see [11, Theorem 1.11]), one can show that the set of normalized entropies of monodromies of the fibrations on the closed manifolds, obtained from  $N$  by Dehn filling all cusps along the slopes not in  $\{-4, \frac{3}{-2}, \frac{1}{-2}, 2\}$ , have no accumulation values  $\leq 2 \log(\frac{3+\sqrt{5}}{2})$ . By using Claim 3.2, one can see that  $a_g$  has to have a boundary slope in  $\{-4, \frac{3}{-2}, \frac{1}{-2}, 2\}$  eventually. Moreover the set of normalized entropies of the monodromies of the fibrations on the closed manifolds obtained from  $N$  by Dehn filling all cusps along the slopes, one of which is in  $\{-4, \frac{3}{-2}, \frac{1}{-2}, 2\}$ , have no accumulation values  $< 2 \log(\frac{3+\sqrt{5}}{2})$ . Then Claim 3.2 leads to Theorem 3.1(1).

**3.2. Sequence  $\{\delta_g^+\}_{g \geq 2}$ .** Let  $\widehat{\mathcal{M}}^+$  be the set of pseudo-Anosov elements of  $\widehat{\mathcal{M}}$  with orientable invariant foliations. (One can use Proposition 2.1(6) to know whether  $\widehat{\phi}_a \in \widehat{\mathcal{M}}$  has orientable invariant foliations or not.) Let  $\widehat{\delta}_g^+$  be the minimum among dilatations of elements in  $\widehat{\mathcal{M}}^+ \cap \text{Mod}(\Sigma_g)$ . (Since  $\widehat{\mathcal{M}}^+ \cap \text{Mod}(\Sigma_g) \neq \emptyset$  for  $g \geq 2$ ,  $\widehat{\delta}_g^+$  is well-defined.) Clearly  $\delta_g \leq \delta_g^+ \leq \widehat{\delta}_g^+$ . The equality  $\delta_g^+ = \widehat{\delta}_g^+$  holds for all  $2 \leq g \leq 8$  except for  $g = 6$ , see Table 5.

**Theorem 3.3** (Theorem 1.5 in [11]).

- (1) *We have  $\lim_{\substack{g \not\equiv 0 \pmod{6} \\ g \rightarrow \infty}} g \log \widehat{\delta}_g^+ = \log\left(\frac{3+\sqrt{5}}{2}\right)$ .*
- (2) *For large  $g$  such that  $g \equiv 2, 4 \pmod{6}$  or  $g \equiv 3 \pmod{10}$  (resp. such that  $g \equiv 1, 5, 7, 9 \pmod{10}$ ),  $\widehat{\delta}_g^+$  is achieved by the monodromy of some  $\Sigma_g$ -bundle over the circle obtained from  $N(\frac{1}{-2})$  (resp.  $N(\frac{3}{-2})$ ) by Dehn filling both cusps.*

Table 6 shows the fibered class  $(x, y, z) \in H_2(N, \partial N)$  which achieves  $\widehat{\delta}_g^+$  for large  $g \not\equiv 0 \pmod{6}$  and the polynomial  $f_{(x,y,z)}(t)$ .

The proof of Theorem 3.3(1) is similar to that of Theorem 3.1(1). The difference is that in the case  $g \equiv 0 \pmod{6}$ , there exist no examples of elements in  $\widehat{\mathcal{M}}^+$  defined on  $\Sigma_g$  which occur as monodromies of fibrations on manifolds obtained from  $N(\frac{1}{-2})$  or  $N(\frac{3}{-2})$  by Dehn filling both cusps. This is the reason why we need the condition  $g \not\equiv 0 \pmod{6}$ .

If we fix any  $\epsilon > 0$  so that  $1.97475 - \epsilon > 2 \log(\frac{3+\sqrt{5}}{2})$ , then for large  $g$  such that  $g \equiv 0 \pmod{6}$ , we have

$$|\chi(\Sigma_g)| \log \widehat{\delta}_g^+ > 1.97475 - \epsilon > 2 \log\left(\frac{3+\sqrt{5}}{2}\right),$$

see [11, Theorem 1.10].

The emphasis is that in the case  $g \equiv 6 \pmod{12}$ , elements of  $\widehat{\mathcal{M}}^+$  provide a new family of pseudo-Anosovs defined on  $\Sigma_g$  with orientable invariant foliations obtained from  $N(-6)$

or  $N(4)$  by Dehn filling both cusps. By using the examples, we obtained the following bounds in [11, Theorem 1.7].

**Theorem 3.4** (Upper bound on  $\delta_g^+$  for  $g \equiv 6 \pmod{12}$ ).

- (1)  $\delta_g^+ \leq \lambda_{(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})}$  if  $g \equiv 6, 30, 42, 54, 78 \pmod{84}$ . The specialization of the Teichmüller polynomial  $P_\Delta$  at  $(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2}) \in S_\gamma(-6)$  is

$$f_{(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})}(t) = (t^{\frac{g}{2}} + 1)(t^{2g} - t^{\frac{3g}{2}} - t^{g+1} + t^g - t^{g-1} - t^{\frac{g}{2}} + 1).$$

- (2)  $\delta_g^+ \leq \lambda_{(g+2, g-2, -\frac{g}{2})}$  if  $g \equiv 18, 66 \pmod{84}$ . The specialization of the Teichmüller polynomial  $P_\Delta$  at  $(g+2, g-2, -\frac{g}{2}) \in S_\gamma(4)$  is

$$f_{(g+2, g-2, -\frac{g}{2})}(t) = (t^{\frac{g}{2}} + 1)(t^{2g} - t^{\frac{3g}{2}} - t^{g+2} + t^g - t^{g-2} - t^{\frac{g}{2}} + 1).$$

The upper bound  $\limsup_{\substack{g \equiv 6 \pmod{12} \\ g \rightarrow \infty}} g \log \delta_g^+ \leq 2 \log \delta(D_5)$  holds, since the ray of

$$\overline{(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})} \in H_2(N_\gamma(-6), \partial N_\gamma(-6)) \text{ (resp. } \overline{(g+2, g-2, -\frac{g}{2})} \in H_2(N_\gamma(4), \partial N_\gamma(4)))$$

converges to the ray of  $\overline{(3, 3, 1)}$  (resp.  $\overline{(2, 2, -1)}$ ) as  $g$  goes to  $\infty$  which achieves

$$\min \text{Ent}(N(-6), \Omega_S) \text{ (resp. } \min \text{Ent}(N(4), \Omega_S)).$$

In particular the projective class of  $(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})$  (resp.  $(g+2, g-2, -\frac{g}{2})$ ) lies on  $\text{int}(\Delta) \cap S_\beta(-6)$  (resp.  $\text{int}(\Delta) \cap S_\beta(4)$ ) and it converges to the projective class of  $(3, 3, 1)$  (resp.  $(2, 2, 1)$ ) as  $g$  goes to  $\infty$ , see Figure 2(2).

Table 1 in [11] exhibits upper bounds of  $\delta_g^+$  for small  $g$  such that  $g \equiv 0 \pmod{6}$  which improves the bound given in [20, 10].

**3.3. Sequences  $\{\delta_{0,n}\}_{n \geq 4}$  and  $\{\delta(D_n)\}_{n \geq 3}$ .** The mapping class group  $\text{Mod}(D_n)$  on an  $n$ -punctured disk  $D_n$  is isomorphic to the subgroup of  $\text{Mod}(\Sigma_{0,n+1})$  consisting of the elements which fix a puncture of  $\Sigma_{0,n+1}$ . (Hence  $\delta(D_n) \geq \delta_{0,n+1}$ .) By using the usual isomorphism  $\Gamma : B_n \rightarrow \text{Mod}(D_n)$  from the  $n$ -braid group  $B_n$  to  $\text{Mod}(D_n)$ , one represents each element of  $\text{Mod}(D_n)$  by an  $n$ -braid.

Let  $\mathcal{N}_n$  be the set of primitive fibered classes  $a \in H_2(N, \partial N)$  such that  $F_a \simeq \Sigma_{0,n}$ . In [12], we ask about which fibered class in  $\mathcal{N}_n$  achieves the minimal dilatation. To give a statement more precisely, let us define an  $m$ -braid  $T_{m,p}$  for  $p \geq 1$  as follows.

$$T_{m,p} = (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^p \sigma_{m-1}^{-2} = (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^{p-1} \sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-2} \sigma_{m-2}^{-1}.$$

If one forgets the 1st strand of  $T_{m,p}$ , one obtains the  $(m-1)$ -braid, call it  $T'_{m,p}$ . Observe that  $\lambda(T'_{m,p}) \leq \lambda(T_{m,p})$  if  $T'_{m,p}$  is pseudo-Anosov. It was shown that the mapping torus  $\mathbb{T}(\Gamma(T_{m,p}))$  is homeomorphic to  $N$  if  $\gcd(m-1, p) = 1$  [12, Corollary 3.2]. Otherwise  $\mathbb{T}(\Gamma(T_{m,p}))$  is toroidal, i.e.  $\Gamma(T_{m,p})$  is reducible [12, Lemma 3.11]. Table 7 describes our result in [12, Theorem 1.1] which answers the above question. For  $n \geq 9$ , the fibered class  $s_n = (x, y, z)$  which achieves the minimal dilatation in  $\mathcal{N}_n$  and its mapping class  $\phi_{s_n}$  are given in the table. (The statement in the case  $4 \leq n \leq 8$  can be found in [12, Theorem 1.1].) Here, we have a remark on the same table (4th column). By Proposition 2.1(1),  $\#(\partial_\alpha F_{s_n}) = 1$  holds. (Also  $\#(\partial_\beta F_{s_n}) = 1$ .) Hence the monodromy  $\Phi_{s_n} : F_{s_n} (\simeq \Sigma_{0,n}) \rightarrow F_{s_n}$  of the fibration associated to  $s_n$  on  $N$  is described by an element in  $\text{Mod}(D_{n-1})$ , and

hence by an  $(n-1)$ -braid. (In this case it turns out that the braid is given by  $T_{n-1,p}$  for some  $p$ .)

We denote by  $T_{(n-1)}$ , the braid  $T_{n-1,*}$  in Table 7(4th column) which represents  $\phi_{s_n}$  for the fibered class  $s_n$ . For example, when  $n = 2k + 1$ ,  $T_{(2k)} = T_{2k,2}$ . The stable foliation  $\mathcal{F}_{s_n}$  has the property such that the boundary component of  $F_{s_n}$  which lies on the torus  $T_\alpha$  has  $x(\neq 1)$  prong, see Proposition 2.1(5). This implies that  $T'_{(n-1)} \in B_{n-2}$  is pseudo-Anosov and  $\lambda(T'_{(n-1)}) = \lambda(T_{(n-1)})$ . One can use both  $(n-2)$ -braids  $T_{(n-2)}$  and  $T'_{(n-1)}$  for upper bounds of  $\delta(D_{n-2})$ , see Table 8(5th column). We would like to point out that  $T'_{(2k)} = T'_{2k,2} \in B_{2k-1}$  is conjugate to the braid called  $\sigma_{k-2,k}$  in [10]. For small  $n$ , our upper bound of  $\delta(D_{n-2})$  is given in Table 9.

The minimal dilatation  $\delta(D_n)$  is determined for all  $3 \leq n \leq 8$  [7, 8, 15, 17]. In these cases, the minimizers “come from”  $N$ . More precisely, the minimal representative  $F_{(x,y,z)}$  of the fibered class  $(x, y, z) \in H_2(N, \partial N)$  in Table 10 is homeomorphic to  $\Sigma_{0,n+2}$ . It turns out that the mapping class  $\phi_{(x,y,z)}$  is of the form  $T_{n+1,p}$  for some  $p$ . Except for  $n = 6$ , the braid  $T'_{n+1,p} \in B_n$  in Table 10(6th column) achieves the minimal dilatation  $\delta(D_n)$ . In the case  $n = 6$ , the braid  $T_{6,2}$  achieves the minimal dilatation  $\delta(D_6)$ .

Observe that  $s_n \in \text{int}(C_\Delta) \cap S_\gamma(\infty)$  and the ray of  $s_n$  converges to the ray of  $[\frac{1}{2}, \frac{1}{2}] = (\frac{1}{2}, \frac{1}{2}, 0) \in \text{int}(\Delta)$  as  $n$  goes to  $\infty$ , see Figure 2(3). By Proposition 2.1(4), we obtain

$$\limsup_{n \rightarrow \infty} n \log \delta(D_n), \limsup_{n \rightarrow \infty} n \log \delta_{0,n} \leq \min \text{Ent}(N) = 2 \log(2 + \sqrt{3}).$$

**3.4. Sequence  $\{\delta_{1,n}\}_{n \geq 1}$ .** Let  $\mathcal{W}_n \subset H_2(N(1), \partial N(1))$  be the set of primitive fibered classes whose minimal representatives are homeomorphic to  $\Sigma_{1,n}$ , see Remark 2.2. In Table 11, one can find the fibered class  $\overline{w_n} = (x, y, z) \in H_2(N_\gamma(1), \partial N_\gamma(1))$  which achieves the minimal dilatation in  $\mathcal{W}_n$ , see [11, Proposition 3.30]. The dilatation of  $w_n \in H_2(N, \partial N)$  is equal to the dilatation of  $\overline{w_n}$ , since  $\mathcal{F}_{w_n}$  has the property such that the boundary components of  $F_{w_n}$  which lie on  $T_\gamma$  has 3 prong, see Proposition 2.1(5). Thus we have

$$\delta_{1,n} \leq \lambda(\overline{w_n}) = \lambda(w_n) = \lambda_{(x,y,z)}.$$

For the polynomial  $f_{(x,y,z)}(t)$  in this case, see Table 11(3rd column).

The ray of  $\overline{w_n} \in H_2(N_\gamma(1), \partial N_\gamma(1))$  converges to the ray of  $(1, 1, -2)$  as  $n$  goes to  $\infty$  which achieves  $\min \text{Ent}(N(1))$ , see Figure 2(4). Thus

$$\limsup_{n \rightarrow \infty} n \log \delta_{1,n} \leq \min \text{Ent}(N(1)) = 2 \log \delta(D_4).$$

Table 12 shows our upper bound of  $\delta_{1,n}$  for small  $n$  due to the brute computation. It turns out that this coincides with the upper bound given by Table 11.

**3.5.  $g > 1$ , Sequence  $\{\delta_{g,n}\}_{n \geq 1}$ .** So far, for the upper bounds of normalized entropies of pseudo-Anosovs, we used the following property of hyperbolic surface bundles over the circle  $M$ : Let  $\Omega$  be a fibered face of  $M$  and let  $\mathcal{D} \subset \text{int}(\Omega)$  be any compact set. Then there exists a constant  $c = c_{\mathcal{D}} > 0$  such that for any fibered class  $a \in \text{int}(C_\Omega)$ , we have  $\text{Ent}(a) = \text{Ent}(\Phi_a) \leq c$  whenever the projective class  $a'$  of  $a$  is in the compact set  $\mathcal{D}$ . However for any fixed  $g \geq 2$ , the same technique doesn't work in order to give an upper bound of  $\delta_{g,n}$  varying  $n$  because of Tsai's result  $\log \delta_{g,n} \asymp \frac{\log n}{n}$ . Her result implies that if there exists a sequence of primitive fibered class classes  $\{a_i\}$  with  $a_i = a_{g,i} \in \text{int}(C_\Omega)$  such that the fiber of the fibration associated to  $a_i$  is a surface of genus  $g$  and  $n_i$  boundary

components with  $n_i \rightarrow \infty$ , then accumulation points of the sequence of projective classes  $\{a'_i\}$  must lie on the boundary of  $\Omega$ .

In [14], we found such a sequence  $\{a_i\} = \{a_{g,i}\}$  of the primitive fibered class  $a_i \in \text{int}(C_\Delta) \cap S_\beta(-1)$  of  $N$  for each  $g \geq 2$  with the best possible asymptotic behavior, i.e.,  $\log \lambda(a_i) = \log \lambda(\Phi_{a_i}) \asymp \frac{\log \|a_i\|}{\|a_i\|}$ . These examples have the property such that the projective class  $a'_i$  goes to a particular point  $(\frac{1}{2}, 1, \frac{1}{2}) \in \partial\Delta$  as  $i$  goes to  $\infty$ , see Figure 2(5). By using the sequence  $\{a_i\}$ , we proved the following.

**Theorem 3.5** ([14]). *Given  $g \geq 2$ , there exists a sequence  $\{n_i\}_{i=0}^\infty$  with  $n_i \rightarrow \infty$  such that  $\limsup_{i \rightarrow \infty} \frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq 2$ . Furthermore, if  $g \geq 2$  enjoys*

$$(*) \quad \gcd(2g+1, s) = 1 \text{ or } \gcd(2g+1, s+1) = 1 \text{ for each } 0 \leq s \leq g,$$

then

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2.$$

For example,  $(*)$  holds for  $g = 4$  since 9 is relatively prime to 1, 2, 4 and 5, but  $(*)$  does not hold for  $g = 7$  because  $\gcd(15, 5) = 5$  and  $\gcd(15, 6) = 3$ . Observe that  $g$  enjoys  $(*)$  if  $2g+1$  is prime. (Hence infinitely many  $g$ 's satisfy  $(*)$ .)

The inequality (2) in Theorem 3.5 improves the upper bound  $\limsup_{n \rightarrow \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2(2g+1)$  (see [14]) obtained from Tsai's examples. Note that this upper bound holds for any  $g \geq 2$ .

#### 4. QUESTIONS AND CONJECTURES

We close with some questions and conjectures about pseudo-Anosovs with the minimal dilatations and their mapping tori.

**Conjecture 4.1** ([11]).

- (1) We have  $\lim_{g \rightarrow \infty} g \log \delta_g = \log(\frac{3+\sqrt{5}}{2})$ . For large  $g$ ,  $\delta_g$  is achieved by the monodromy of some  $\Sigma_g$ -bundle over the circle obtained from either  $N(\frac{3}{-2})$  or  $N(\frac{1}{-2})$  by Dehn filling both cusps.
- (2) We have  $\lim_{\substack{g \not\equiv 0 \pmod{6} \\ g \rightarrow \infty}} g \log \delta_g^+ = \log(\frac{3+\sqrt{5}}{2})$ . For large  $g$  such that  $g \not\equiv 0 \pmod{6}$ ,  $\delta_g^+$  is achieved by the monodromy of some  $\Sigma_g$ -bundle over the circle obtained from  $N(\frac{3}{-2})$  or  $N(\frac{1}{-2})$  by Dehn filling both cusps.

**Conjecture 4.2** ([12]).

- (1)  $\delta(D_{2k-1}) = \lambda(T'_{2k,2})$  for  $k \geq 5$ .
- (2)  $\delta(D_{4k}) = \lambda(T'_{4k+1,2k-1})$  for  $k \geq 3$ .
- (3)  $\delta(D_{10}) = \lambda(T'_{10,2})$ , and  $\delta(D_{8k+2}) = \lambda(T'_{8k+3,2k+1})$  for  $k \geq 2$ .
- (4)  $\delta(D_{8k+6}) = \lambda(T'_{8k+7,2k+1})$  for  $k \geq 1$ .

**Conjecture 4.3** ([11]). We have  $\lim_{n \rightarrow \infty} n \log \delta_{1,n} = 2 \log \delta(D_4)$ . For large  $n$ ,  $\delta_{1,n}$  is achieved by the monodromy of a fibration on  $N(1)$ .

**Question 4.4** ([14]). Can one eliminate the condition  $(*)$  in Theorem 3.5? i.e., given  $g \geq 2$ , does  $\limsup_{n \rightarrow \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2$  hold?

Finally, we ask about questions related to the finiteness theorem for small dilatation pseudo-Anosov homeomorphisms [5, 3]. Given a pseudo-Anosov  $\Phi : \Sigma \rightarrow \Sigma$ , let  $\Sigma^\circ \subset \Sigma$  be the surface obtained by removing all the singularities of the stable foliation for  $\Phi$ , and  $\Phi|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \Sigma^\circ$  denotes the restriction of  $\Phi$  to  $\Sigma^\circ$ . Observe that  $\lambda(\Phi) = \lambda(\Phi|_{\Sigma^\circ})$ . The finiteness theorem implies that the following sets are finite.

$$\begin{aligned}\mathcal{U} &= \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = \Sigma_g \text{ such that } \lambda(\Phi) = \delta_g, g \geq 2\}, \\ \mathcal{U}_{\text{braid}} &= \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = D_n \text{ such that } \lambda(\Phi) = \delta(D_n), n \geq 3\}, \\ \mathcal{U}_{g=1} &= \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = \Sigma_{1,n} \text{ such that } \lambda(\Phi) = \delta_{1,n}, n \geq 1\}.\end{aligned}$$

We know that  $N \in \mathcal{U} \cap \mathcal{U}_{\text{braid}} \cap \mathcal{U}_{g=1}$ . Since pseudo-Anosov mapping classes with the smallest known dilatations defined on either  $\Sigma_g$ ,  $D_n$  or  $\Sigma_{1,n}$  come from  $N$ , we ask:

**Question 4.5.** *It is true that  $\mathcal{U} = \mathcal{U}_{\text{braid}} = \mathcal{U}_{g=1} = \{N\}$ ?*

On the other hand, by the fact that given  $g \geq 2$ ,  $\log \delta_{g,n} \asymp \frac{\log n}{n}$ , one can not appeal to the finiteness theorem for the following set  $\mathcal{U}_g$  for  $g \geq 2$ .

$$\mathcal{U}_g = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = \Sigma_{g,n} \text{ such that } \lambda(\Phi) = \delta_{g,n}, n \geq 1\}.$$

The examples which provide the upper bound in Theorem 3.5 are monodromies of fibrations on manifolds obtained from the single manifold  $N$  by Dehn fillings. For this reason, we would like to ask:

**Question 4.6.** *Is there any  $g \geq 2$  such that  $\mathcal{U}_g$  is a finite set?*

## 5. TABLES

TABLE 3. fibered class  $(x, y, z) \in H_2(N, \partial N)$  which achieves  $\widehat{\delta}_g$  for large  $g$ , see [11, Theorem 1.4, Remark 3.18]. [notice that  $(x, y, z)$  is in either  $S_\beta(\frac{3}{-2})$  or  $S_\beta(\frac{1}{-2})$ .]

$g$	$(x, y, z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$
0, 1, 5, 6 (mod 10)	$(2g + 5, 2g + 6, g + 4) \in S_\beta(\frac{3}{-2})$	$(t^{g+3} + 1)(t^{2g+4} - t^{g+3} - t^{g+2} - t^{g+1} + 1)$
7, 9 (mod 10)	$(2g + 6, 2g + 8, g + 6) \in S_\beta(\frac{3}{-2})$	$(t^{g+4} + 1)(t^{2g+4} - t^{g+4} - t^{g+2} - t^g + 1)$
[3 (mod 10)]		
3, 13 (mod 30)	$(g + 1, 2g + 8, 3) \in S_\beta(\frac{1}{-2})$	$(t^{g+4} + 1)(t^{2g+2} - t^{g+4} - t^{g+1} - t^{g-2} + 1)$
23 (mod 30)	$(g + 1, 2g + 4, 1) \in S_\beta(\frac{1}{-2})$	$(t^{g+2} + 1)(t^{2g+2} - t^{g+2} - t^{g+1} - t^g + 1)$
[8 (mod 10)]		
8 (mod 30)	$(g + 1, 2g + 4, 1) \in S_\beta(\frac{1}{-2})$	$(t^{g+2} + 1)(t^{2g+2} - t^{g+2} - t^{g+1} - t^g + 1)$
18, 28 (mod 30)	$(g + 1, 2g + 8, 3) \in S_\beta(\frac{1}{-2})$	$(t^{g+4} + 1)(t^{2g+2} - t^{g+4} - t^{g+1} - t^{g-2} + 1)$



TABLE 4. upper bounds of  $\delta_g$  for small  $g$ . [see also [9, 1, 13].]

$g$	$(x, y, z) \in H_2(N, \partial N)$	$(\delta_g \leq) \lambda_{(x,y,z)} \approx$
3	$(4, 14, 3) \in S_\beta(\frac{1}{-2})$	1.4012
4	$(5, 16, 3) \in S_\beta(\frac{1}{-2})$	1.2612
5	$(13, 12, 5) \in S_\beta(\frac{3}{-2})$	1.1487
6	$(15, 14, 6) \in S_\beta(\frac{3}{-2})$	1.1287
7	$(16, 14, 5) \in S_\beta(\frac{3}{-2})$	1.1154
8	$(17, 18, 7) \in S_\beta(\frac{4}{-3})$	1.1040
9	$(20, 18, 7) \in S_\beta(\frac{3}{-2})$	1.0928
10	$(23, 22, 10) \in S_\beta(\frac{3}{-2})$	1.0837
11	$(25, 24, 11) \in S_\beta(\frac{3}{-2})$	1.0770
12	$(25, 22, 8) \in S_\beta(\frac{3}{-2})$	1.0726
13	$(27, 21, 8) \in S_\beta(\frac{5}{-3})$	1.0716
14	$(29, 26, 10) \in S_\beta(\frac{3}{-2})$	1.0629
15	$(33, 32, 15) \in S_\beta(\frac{3}{-2})$	1.0583
16	$(35, 34, 16) \in S_\beta(\frac{3}{-2})$	1.0549
17	$(36, 34, 15) \in S_\beta(\frac{3}{-2})$	1.0522
18	$(19, 44, 3) \in S_\beta(\frac{1}{-2})$	1.0525
19	$(40, 38, 17) \in S_\beta(\frac{3}{-2})$	1.0470
20	$(43, 42, 20) \in S_\beta(\frac{3}{-2})$	1.0447

TABLE 5. fibered class  $(x, y, z) \in H_2(N, \partial N)$  which achieves  $\delta_g^+$  for small  $g$ .

$g$	$(x, y, z) \in H_2(N, \partial N)$	$\delta_g^+ = \lambda_{(x,y,z)} \approx$	minimal polynomial (a factor of $f_{(x,y,z)}(t)$ )
2	$(2, 6, 1) \in S_\beta(\frac{1}{-2})$	1.7220 [26]	$t^4 - t^3 - t^2 - t + 1$
3	$(4, 14, 3) \in S_\beta(\frac{1}{-2})$	1.4012 [16]	$t^6 - t^4 - t^3 - t^2 + 1$
4	$(4, 10, 1) \in S_\beta(\frac{1}{-2})$	1.2806 [16]	$t^8 - t^5 - t^4 - t^3 + 1$
5	$(18, 22, 15) \in S_\beta(\frac{3}{-2})$	1.1762 [16]	$t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$
7	$(20, 22, 13) \in S_\beta(\frac{3}{-2})$	1.1154 [16, 1, 13]	$t^{14} + t^{13} - t^9 - t^8 - t^7 - t^6 - t^5 + t + 1$
8	$(8, 18, 1) \in S_\beta(\frac{1}{-2})$	1.1287 [16, 9]	$t^{16} - t^9 - t^8 - t^7 + 1$

TABLE 6. fibered class  $(x, y, z) \in H_2(N, \partial N)$  which achieves  $\widehat{\delta}_g^+$  for large  $g \not\equiv 0 \pmod{6}$ , see [11, Theorem 1.5]. [notice that  $(x, y, z)$  is in either  $S_\beta(\frac{3}{-2})$  or  $S_\beta(\frac{1}{-2})$ .]

$g$	$(x, y, z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$
7, 9 (mod 10)	$(2g + 6, 2g + 8, g + 6) \in S_\beta(\frac{3}{-2})$	$(t^{g+4} + 1)(t^{2g+4} - t^{g+4} - t^{g+2} - t^g + 1)$
1, 5 (mod 10)	$(2g + 8, 2g + 12, g + 10) \in S_\beta(\frac{3}{-2})$	$(t^{g+6} + 1)(t^{2g+4} - t^{g+6} - t^{g+2} - t^{g-2} + 1)$
[3 (mod 10)]		
3, 13 (mod 30)	$(g + 1, 2g + 8, 3) \in S_\beta(\frac{1}{-2})$	$(t^{g+4} + 1)(t^{2g+2} - t^{g+4} - t^{g+1} - t^{g-2} + 1)$
23 (mod 30)	$(g + 1, 2g + 4, 1) \in S_\beta(\frac{1}{-2})$	$(t^{g+2} + 1)(t^{2g+2} - t^{g+2} - t^{g+1} - t^g + 1)$
2, 4 (mod 6)	$(g, 2g + 2, 1) \in S_\beta(\frac{1}{-2})$	$(t^{g+1} + 1)(t^{2g} - t^{g+1} - t^g - t^{g-1} + 1)$

TABLE 7. for  $n \geq 9$ , fibered class  $s_n$  which achieves the minimal dilatation in  $\mathcal{N}_n$  and its mapping class  $\phi_{s_n}$ , see [12, Theorem 1.1]. [notice that  $s_n \in S_\gamma(\infty)$ .]

$n$	$s_n = (x, y, z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$	$\phi_{s_n}$
$2k+1$	$(k-1, k, 0) \in S_\gamma(\infty)$	$t^{2k-1} - 2(t^{k-1} + t^k) + 1$	$T_{2k,2}$
$4k+2$	$(2k+1, 2k-1, 0) \in S_\gamma(\infty)$	$t^{4k} - 2(t^{2k-1} + t^{2k+1}) + 1$	$T_{4k+1,2k-1}$
$8k+4$	$(4k-1, 4k+3, 0) \in S_\gamma(\infty)$	$t^{8k+2} - 2(t^{4k-1} + t^{4k+3}) + 1$	$T_{8k+3,2k+1}$
$8(k+1)$	$(4k+5, 4k+1, 0) \in S_\gamma(\infty)$	$t^{8k+6} - 2(t^{4k+1} + t^{4k+5}) + 1$	$T_{8k+7,2k+1}$

TABLE 8. upper bounds of  $\delta(D_{n-2})$ , see [12, Corollary 4.1]. [see also [10, 25].]

$n$	$s_n = (x, y, z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$	filling	braid $\in B_{n-2}$
$2k+1$	$(k-1, k, 0) \in S_\gamma(\infty)$	$t^{2k-1} - 2(t^{k-1} + t^k) + 1$	$\alpha$	$T'_{2k,2}$
$4k+2$	$(2k+1, 2k-1, 0) \in S_\gamma(\infty)$	$t^{4k} - 2(t^{2k-1} + t^{2k+1}) + 1$	$\alpha$	$T'_{4k+1,2k-1}$
$8k+4$	$(4k-1, 4k+3, 0) \in S_\gamma(\infty)$	$t^{8k+2} - 2(t^{4k-1} + t^{4k+3}) + 1$	$\alpha$	$T'_{8k+3,2k+1}$
$8(k+1)$	$(4k+5, 4k+1, 0) \in S_\gamma(\infty)$	$t^{8k+6} - 2(t^{4k+1} + t^{4k+5}) + 1$	$\alpha$	$T'_{8k+7,2k+1}$

TABLE 9. upper bounds of  $\delta(D_{n-2})$  for small  $n$ . [see also [10, 25].]

$n$	$(x, y, z) \in H_2(N, \partial N)$	$(\delta(D_{n-2}) \leq) \lambda_{(x,y,z)} \approx$	filling	braid $\in B_{n-2}$
11	$(4, 5, 0) \in S_\gamma(\infty)$	1.3437	$\alpha$	$T'_{10,2}$
12	$(4, 5, 0) \in S_\gamma(\infty)$	1.3437	none	$T_{10,2}$
13	$(5, 6, 0) \in S_\gamma(\infty)$	1.2724	$\alpha$	$T'_{12,2}$
14	$(7, 5, 0) \in S_\gamma(\infty)$	1.2514	$\alpha$	$T'_{13,5}$
15	$(6, 7, 0) \in S_\gamma(\infty)$	1.2257	$\alpha$	$T'_{14,2}$
16	$(9, 5, 0) \in S_\gamma(\infty)$	1.2225	$\alpha$	$T'_{15,3}$
17	$(7, 8, 0) \in S_\gamma(\infty)$	1.1926	$\alpha$	$T'_{16,2}$
18	$(9, 7, 0) \in S_\gamma(\infty)$	1.1812	$\alpha$	$T'_{17,7}$
19	$(8, 9, 0) \in S_\gamma(\infty)$	1.1680	$\alpha$	$T'_{18,2}$
20	$(7, 11, 0) \in S_\gamma(\infty)$	1.1643	$\alpha$	$T'_{19,5}$
21	$(9, 10, 0) \in S_\gamma(\infty)$	1.1490	$\alpha$	$T'_{20,2}$
22	$(11, 9, 0) \in S_\gamma(\infty)$	1.1419	$\alpha$	$T'_{21,9}$
23	$(10, 11, 0) \in S_\gamma(\infty)$	1.1338	$\alpha$	$T'_{22,2}$
24	$(13, 9, 0) \in S_\gamma(\infty)$	1.1307	$\alpha$	$T'_{23,5}$
25	$(11, 12, 0) \in S_\gamma(\infty)$	1.1215	$\alpha$	$T'_{24,2}$
26	$(13, 11, 0) \in S_\gamma(\infty)$	1.1166	$\alpha$	$T'_{25,11}$
27	$(12, 13, 0) \in S_\gamma(\infty)$	1.1112	$\alpha$	$T'_{26,2}$
28	$(11, 15, 0) \in S_\gamma(\infty)$	1.1086	$\alpha$	$T'_{27,7}$
29	$(13, 14, 0) \in S_\gamma(\infty)$	1.1025	$\alpha$	$T'_{28,2}$
30	$(15, 13, 0) \in S_\gamma(\infty)$	1.0990	$\alpha$	$T'_{29,13}$
31	$(14, 15, 0) \in S_\gamma(\infty)$	1.0951	$\alpha$	$T'_{30,2}$
32	$(17, 13, 0) \in S_\gamma(\infty)$	1.0930	$\alpha$	$T'_{31,7}$

TABLE 10. fibered class  $(x, y, z) \in H_2(N, \partial N)$  which achieves  $\delta(D_n)$  for small  $n$ , see [12, Section 4.1]. [for the minimal polynomial of  $\delta(D_n)$ , see the 4th column.]

$n$	$(x, y, z) \in H_2(N, \partial N)$	$\delta(D_n) = \lambda_{(x,y,z)} \approx$	minimal polynomial	filling	braid $\in B_n$
3	$(2, 1, 0) \in S_\gamma(\infty)$	$\frac{3+\sqrt{5}}{2}$ [7]	$t^2 - 3t + 1$	$\alpha$	$T'_{4,1}$
4	$(3, 1, 0) \in S_\gamma(\infty)$	2.2966 [15]	$t^4 - 2t^3 - 2t + 1$	$\alpha$	$T'_{5,1}$
5	$(2, 3, 0) \in S_\gamma(\infty)$	1.7220 [8]	$t^4 - t^3 - t^2 - t + 1$	$\alpha$	$T'_{6,2}$
6	$(2, 3, 0) \in S_\gamma(\infty)$	1.7220 [17]	$t^4 - t^3 - t^2 - t + 1$	none	$T_{6,2}$
7	$(3, 4, 0) \in S_\gamma(\infty)$	1.4655 [17]	$t^3 - t^2 - 1$	$\alpha$	$T'_{8,2}$
8	$(5, 3, 0) \in S_\gamma(\infty)$	1.4134 [17]	$t^8 - 2t^5 - 2t^3 + 1$	$\alpha$	$T'_{9,5}$

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TABLE 11. fibered class  $\overline{w_n} \in H_2(N_\gamma(1), \partial N_\gamma(1))$  which achieves the minimal dilatation in  $\mathcal{W}_n$ , see [11, Proposition 3.30]. [notice that  $w_n \in S_\gamma(1)$ .]

$n$	$w_n = (x, y, z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$	filling
2	$(1, 1, -2) \in S_\gamma(1)$	$t^4 - 2t^3 - 2t + 1$	$\gamma$
$2k - 1$	$(k, k - 1, -2k + 1) \in S_\gamma(1)$	$t^{4k-2} - t^{3k-1} - t^{3k-2} - t^k - t^{k-1} + 1$	$\gamma$
$4k$	$(2k + 1, 2k - 1, -4k) \in S_\gamma(1)$	$t^{8k} - t^{6k+1} - t^{6k-1} - t^{2k+1} - t^{2k-1} + 1$	$\gamma$
$4k + 2$	$(2k + 3, 2k - 1, -4k - 2) \in S_\gamma(1)$	$t^{8k+4} - t^{6k+5} - t^{6k+1} - t^{2k+3} - t^{2k-1} + 1$	$\gamma$

TABLE 12. upper bounds of  $\delta_{1,n}$  for small  $n$ .

$n$	$(x, y, z) \in H_2(N, \partial N)$	$(\delta_{1,n} \leq) \lambda_{(x,y,z)} \approx$	filling
2	$(1, 1, -2) \in S_\gamma(1)$	2.2966	$\gamma$
3	$(2, 1, -3) \in S_\gamma(1)$	1.7816	$\gamma$
4	$(3, 1, -4) \in S_\gamma(1)$	1.5823	$\gamma$
5	$(3, 2, -5) \in S_\gamma(1)$	1.4012	$\gamma$
6	$(5, 1, -6) \in S_\gamma(1)$	1.4012	$\gamma$
7	$(4, 3, -7) \in S_\gamma(1)$	1.2703	$\gamma$
8	$(5, 3, -8) \in S_\gamma(1)$	1.2369	$\gamma$
9	$(5, 4, -9) \in S_\gamma(1)$	1.2039	$\gamma$
10	$(7, 3, -10) \in S_\gamma(1)$	1.1932	$\gamma$
11	$(6, 5, -11) \in S_\gamma(1)$	1.1637	$\gamma$
12	$(7, 5, -12) \in S_\gamma(1)$	1.1502	$\gamma$
13	$(7, 6, -13) \in S_\gamma(1)$	1.1367	$\gamma$
14	$(9, 5, -14) \in S_\gamma(1)$	1.1301	$\gamma$
15	$(8, 7, -15) \in S_\gamma(1)$	1.1174	$\gamma$
16	$(9, 7, -16) \in S_\gamma(1)$	1.1101	$\gamma$
17	$(9, 8, -17) \in S_\gamma(1)$	1.1028	$\gamma$
18	$(11, 7, -18) \in S_\gamma(1)$	1.0986	$\gamma$
19	$(10, 9, -19) \in S_\gamma(1)$	1.0915	$\gamma$
20	$(11, 9, -20) \in S_\gamma(1)$	1.0870	$\gamma$

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